GRAPH THEORY COURSE NOTES
WINTER 2011 PROFESSOR DAPHNE LIU

ADAM MARKS

1. Definitions

A *graph* is a finite nonempty set V of vertices together with a set of 2-element subsets of V called edges. $G = (V, E)$

The *order* of a graph is the number of vertices.

The *size* of a graph is the number of edges.

The ends of an edge are *incident* with the edge, and vice versa.

Two vertices which are incident with a common edge are *adjacent*, as are two edges which are incident with a common vertex.

The *degree* of a vertex is the number of vertices adjacent to it.

$\delta(G)$ is the minimum degree of any vertex of G.

$\Delta(G)$ is the maximum degree of any vertex of G.

The *(open) neighborhood* of v, $N(v)$, is the set of vertices adjacent to v.

The *closed neighborhood* of v, $N[v]$, is the open neighborhood of v plus v itself.

The *open neighborhood* of a subset S of V is the union of the open neighborhoods of the vertices in S.

$$N(S) = \bigcup_{v \in S} N(v)$$

A graph H is *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Date: March 13, 2011.

A *proper subgraph* of G is a subgraph H with $H \neq G$.

A *spanning subgraph* of G is a subgraph H with $V(H) = V(G)$.

For a nonempty subset S of V, the subgraph *induced* by S, $G[S]$, is the subgraph whose vertex set is S and whose edge set is the set of those edges of G that have both ends in S.

For a nonempty subset F of E, the subgraph *induced* by F, $G[F]$, is the subgraph whose vertex set is the set of ends of edges in F and whose edge set is F.

For two vertices u and v in G, a *uv walk* is a sequence of vertices in G beginning at u and ending at v such that consecutive vertices are adjacent.

The *length* of a walk is the number of edges in it.

A *trail* is a walk in which no edge is repeated.

A *path* is a walk in which no vertex is repeated.

A walk whose initial and terminal vertices are distinct is *open*, otherwise it is *closed*.

A *circuit* is a nontrivial closed walk in which no edge is repeated.

A *cycle* is a circuit for which all but the initial and terminal vertices are distinct.

Two vertices are *connected* in G if there exists a path in G between them.

A graph is *connected* if there exists a path between every pair of vertices.

A *component* is a maximal connected subgraph.

The *distance* $d(u, v)$ from a vertex u to a vertex v in a connected graph is the minimum of the lengths of the uv paths.

A *geodesic* is a uv path of length $d(u, v)$.

The *eccentricity* $e(v)$ of a vertex v in a connected graph is the distance between v and a vertex farthest from v.

$$e(v) = max\{d(u, v) : u, v \in V\}$$

The *diameter* $diam(G)$ is the greatest eccentricity among vertices of G, or the greatest distance between any two vertices.

$$diam(G) = max\{e(v) : v \in V\} = max\{d(u, v) : u, v \in V\}$$

The *radius* $rad(G)$ is the smallest eccentricity among the vertices of G.

$$rad(G) = min\{e(v) : v \in V\}$$

A *central vertex* is a vertex with $e(v) = rad(G)$.

The *center* $Cen(G)$ is the subgraph induced by the central vertices of G.

$$Cen(G) = G[\{v : e(v) = rad(G)\}]$$

A *peripheral vertex* of G is a vertex v with $e(v) = diam(G)$.

Two vertices u and v with $d(u, v) = diam(G)$ are *antipodal*, and each is necessarily a peripheral vertex.

If every vertex of G is a central vertex, then $Cen(G) = G$ and G is *self-centered*.

A *homomorphism* from G to H is a function $\phi : V(G) \longrightarrow V(H)$ that maps adjacent vertices in G to adjacent vertices in H. The images of nonadjacent vertices may be either nonadjacent, adjacent, or equal.

G and H are *homomorphic* if there exists a homomorphism from G to H, denoted $G \longrightarrow H$.

An *isomorphism* from G to H is a bijective function $\phi : V(G) \longrightarrow V(H)$ such that two vertices are adjacent in G if and only if their images are adjacent in H.

G and H are *isomorphic* if there exists an isomorphism from G to H, denoted $G \cong H$.

A graph is *complete* if every two distinct vertices are adjacent. The complete graph of order n is denoted K_n.

A graph G is *bipartite* if it is possible to partition V into two subsets U and W, called partite sets, such that every edge of G joins a vertex in U and W.

A bipartite graph with partite sets U and W is a *complete bipartite graph* if every vertex of U is adjacent to every vertex of W. If the partite sets contain s and t vertices, the graph is denoted by $K_{s,t}$.

If a graph G is *regular*, all of the vertices have the same degree. If the every vertex has degree r, G is called r-regular.

A *cubic* graph is 3-regular.

The Petersen graph is:

A nontrivial graph G is *multipartite*, or k-partite if it is possible to partition V into k partite sets $V_1, V_2, \ldots, V_k$, $k \geq 2$, such that every edge of G joins vertices in different partite sets.

A k-partite graph in which for all i and j with $1 \leq i, j \leq k$ and $i \neq j$, for all $u \in V_i$ and $v \in V_j$, u and v are adjacent, is called a *complete k-partite graph* and denoted $K_{n_1, n_2, \ldots n_k}$ where $|V_i| = n_i$, where V_i are the partite sets.

The *hypercube* Q_n has as each vertex a binary string of length n, and each pair of vertices is adjacent if they are equal at all components except for one.

The *complement* $\overline{G}$ of G is that graph whose vertex set is $V(G)$ and where uv is an edge of $\overline{G}$ if and only if uv is not an edge in G.

The *line graph* $L(G)$ of a nonempty graph G is that graph whose vertex set is $E(G)$, and two vertices e and f of $L(G)$ are adjacent if and only if e and f are adjacent edges in G.

A *tree* is a connected acyclic graph.

A *forest* is an acyclic graph.

A *leaf* is a vertex of degree 1.

A vertex v is a *cut-vertex* if $G - v$ has more components than G.

A subset S of V is a *vertex-cut* if $G - S$ has more components than G.

An edge e is a *cut-edge* or *bridge* if $G - e$ has more components than G.

A subset F of E is an *edge-cut* if $G - F$ has more components than G.

A graph is *non-separable* if has no cut-vertices.

A *spanning tree* of G is a subgraph that is a tree with the same vertex set as G.

A *block* is a maximal non-separable subgraph.

The *vertex-connectivity*, $\kappa(G)$ is the minimum number of vertices whose removal yields either a disconnected or trivial graph.

The *edge-connectivity*, $\lambda(G)$ is the minimum number of edges whose removal yields a disconnected graph.

A subset S of V is called an *independent set (of vertices)* of G if no two vertices of S are adjacent in G.

A subset F of E is called an *independent set of edges* in G if no two edges in E share a common end in G.

A *matching* is a set of independent edges.

M is a *maximum matching* if G has no matching M' with $|M'| > |M|$.

M is a *maximal matching* if M is not a proper subset of another matching.

M is a *perfect matching* if every vertex of G is incident with some edge of M.

An *M-augmenting path* in G is a path in which the edges alternate between $E \setminus M$ and M and its end vertices are M-unsaturated.

A collection of finite nonempty sets $S_1, S_2, \ldots, S_n$ has a *system of distinct representatives* if there exist n distinct elements $x_1, x_2, \ldots, x_n$ such that $x_i \in S_i$ for $1 \leq i \leq n$.

The number of vertices in a maximum set of independent vertices is $\alpha(G)$, the *independence number* of G.

The number of edges in a maximum set of independent edges is $\alpha'(G)$, the *edge independence number* of G.

The maximum number l such that K_l is a subgraph of G is $\omega(G)$, the *clique number* of G.

A *factor* of G is a spanning subgraph of G.

A *k-factor* of G is a k-regular spanning subgraph of G. Note that a 1-factor is a perfect matching and a 2-factor is a union of disjoint cycles.

A *factorization* $\mathcal{F}$ of G is a collection of factors of G such that every edge of G belongs to exactly one factor of $\mathcal{F}$.

An open trail is a *Eularian trail*.

A closed trail is a *Eularian circuit*.

A graph with a Eularian circuit is a *Eularian graph*.

A spanning path is a *Hamiltonian path*.

A spanning cycle is a *Hamiltonian cycle*.

A graph with a Hamiltonian cycle is a *Hamiltonian graph*.

If there exists a factorization $\mathcal{F}$ of G such that each factor in $\mathcal{F}$ is a Hamiltonian cycle of G, then G is *Hamiltonian factorable*.

$k_o(G)$ is the number of components of G that have an odd number of vertices.

The *closure* of a graph G with order n, $cl(G)$, is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair remains.

A *planar* graph is a graph that can be drawn in a plane without crossing edges.

A *plane* graph is a graph with a given planar embedding.

The *crossing number* of a graph is the minimum number of crossing edges for a graph drawn in a plane.

A plane graph divides the plane into *regions*.

The *exterior region* of plane graph is the region outside every cycle.

The *boundary* of a region of a plane graph is the subgraph incident with the region.

A planar graph is *maximal planar* if the addition of any edge joining nonadjacent vertices results in a nonplanar graph.

Given a plane graph G, the *dual* of G, G^* is defined as follows: corresponding to each region r of G there is a vertex r^* of G^*, and corresponding to each edge e of G there is an edge e^* of G^*; two vertices r^* and s^* are joined by the edge e^* in G^* if and only if their corresponding faces r and s are separated by the edge e in G.

In a planar graph with Hamiltonian cycle C, any edge of G not on C is a *chord*.

A *subdivision* of a graph is made by replacing zero or more edges by a path.

An edge of a graph is said to be *contracted* if it is deleted and its ends are identified.

H is a *minor* of G if H is equal to G or H can be obtained from G by a succession of edge contractions.

A *proper k-coloring* of G is a function $f : V \longrightarrow \{1, 2, \ldots, k\}$ such that $f(u) \neq f(v)$ if $u \sim v$.

A graph is *k-colorable* if there exists a proper k-coloring.

The *chromatic number* of G, $\chi(G)$, is the smallest k for which G admits a k-coloring.

The *clique size* of G, $\omega(G)$, is the largest integer t for which K_t is a subgraph of G.

The *union* $G \cup H$ of two vertex-disjoint graphs G and H is the graph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$.

The *join* $G + H$ of two vertex-disjoint graphs G and H is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$, that is every vertex in G is adjacent to every vertex in H.

The *Mycielski graph* of G, $\mu(G)$ is constructed by:
- $V(\mu(G)) = V(G) \cup V^*(G) \cup \{u^*\}$, where V^* is a copy of V with v' called the *twin* of v for every $v \in V(G)$, and u^* called the *root*.
- $E(\mu(G)) = E(G) \cup \{uv' : uv \in E(G)\} \cup \{v'u^* : v \in V(G)\}$.

A graph that contains no 3-cycles is *triangle-free*.

For two positive integers m, n with $m \geq 2n$, the *Kneser graph* $KG(m,n)$ is defined by:

- $V = \{A : \text{A is an } n\text{-element subset of } \{1, 2, \ldots, m\}\}$
- $A \sim B \Leftrightarrow A \cap B = \emptyset$

2. Theorems

The First Theorem of Graph Theory [1.1] The total degree of the vertices is equal to twice the number of edges. That is, if G is a graph of size m then

$$\sum_{v \in V} deg(v) = 2m.$$

Proof. When summing the total degree of the vertices, every edge is counted twice, once for each incident vertex.

Corollary 1.2 Every graph has an even number of odd vertices.
Proof. Let X be the even vertices and Y the odd. By the first theorem of graph theory, $\sum degx + \sum degy = 2m$. Since the first and third terms are even, the second must also be even.

Triangle Inequality For all $u, v, w \in G$, $d(u,v) \leq d(u,w) + d(w,v)$.

Theorem 1.4 For every nontrivial connected graph G,

$$rad(G) \leq diam(G) \leq 2rad(G).$$

Proof. Choose u, v on the periphery and w in the center. Then by triangle inequality, $diam(G) = d(u,v) \leq d(u,w) + d(w,v) \leq 2rad(G)$.

Theorem An odd cycle is not bipartite.

Theorem 1.10 A nontrivial graph is bipartite if and only if it contains no odd cycles.

Problem 1.12 If G is a self-complementary graph of order n then $n \equiv 0$ or 1 (mod 4).
Proof. $|E(G)| = m$ and $|E(\overline{G})| = \binom{n}{2} - m$. Then $m = \frac{n(n-1)}{2}$ and so $n(n-1) \equiv 0$ mod 4. Thus $n \equiv 0, 1$ mod 4.

Theorem All of these are equivalent:
i) G is self-centered
ii) $diam(G) = rad(G)$
iii) $e(v) = diam(G) = rad(G)$ for all $v \in V$

Corollary 2.8 Every tree contains at least two leaves.
Proof. Choose a path $P = v_0 v_1 \ldots v_k$ of max length. Suppose $deg(v_0) \neq 1$. Then $\exists x$ off P, but then $x v_0 \ldots v_k$ is a path of length $k+1$, a contradiction. Similarly for the other endpoint of P.

Theorem Every connected graph contains a spanning tree.
Proof [algorithm for finding a spanning tree: keep adding vertices adjacent to any vertex already chosen so as to not create a cycle.

Theorem 2.9 For every tree, $|E| = |V| - 1$.
Proof. [By induction on n, consider leaf. Or by Euler Id.]

Theorem 2.3 An edge e in G is a cut-edge if and only if e is not on any cycle of G.

Theorem If G is disconnected, then $\overline{G}$ is connected.
Proof. Let $a, b \in \overline{G}$. Let $V_1, \ldots, V_k$ be the components of G, and WLOG assume $a \in V_1$ and $b \in V_k$. Let $v_1 = a$, $v_k = b$ and choose $v_2, \ldots, v_{k-1}$ vertices in the respective components V_i. Then $v_1 \ldots v_k$ is an ab path in $\overline{G}$, since each pair v_i, v_{i+1} is disconnected in G and thus connected in $\overline{G}$.

Theorem Every tree is bipartite.
Proof. Draw the tree in tiers starting with any vertex as the root. Then the alternating tiers are bipartite sets.

Theorem In a tree, every edge is a bridge.

Theorem If e is a bridge in G, then $comp(G-e) = comp(G) + 1$.

Theorem 2.16 $\kappa(G) \leq \lambda(G) \leq \delta(G)$.
Proof. [For first part, consider a minimum edge-cut; the vertices incident to it in one component are a vertex-cut.]

Theorem Ex 2.7 If G is a graph with order $n \geq 3$ such that $deg(u) + deg(v) \geq n$ for every pair u, v of nonadjacent vertices, then G is nonseparable.
Proof. Suppose x is a cut-vertex, and let W be a component of $G - x$ and R be the remainder of G. Choose $u \in W$ and $v \in R$ such that $u \sim x$ and $v \sim x$ in G. By hypothesis in G there are $n-2$ edges distributed among (but not between) u and v but only $n-3$ vertices in G besides u, v, x. Thus there must be some edge between u and R or between v and W, a contradiction.

Theorem For any vertices u, v in a tree T, there exists a unique uv path.
Proof. [Consider two distinct paths, find a cycle.]

Theorem If T is a tree and $\overline{T}$ is also a tree, then $T \cong P_1$ or P_4, and $T \cong \overline{T}$.

Berge's Theorem A matching M is maximum in G if and only if G has no M-augmenting path.

Hall's Theorem Let G be a bipartite graph with partite sets U and W where $|U| \leq |W|$. Then there exists a matching M that saturates all vertices in U if and only if $|N(S)| \geq |S|$ for all $S \subseteq U$.

Marriage Theorem Every r-regular bipartite graph with $r \geq 1$ has a perfect matching.
Proof [apply Hall's thm]

Theorem K_{2n+1} is Hamiltonian factorable.

Daniel's Homework Lemma If a tree T contains at least 3 leaves, then $T \ncong \overline{T}$.

Mathew's Homework Theorem If G is a graph of order n and for all $u, v \in V$, $u \nsim v$, $deg(u) + deg(v) \geq n$, then $diam(G) \leq 2$.

Tutte's Theorem A nontrivial graph G contains a perfect matching if and only if for all $S \subseteq V$, $k_o(G - S) \leq |S|$.

Petersen's Theorem Every bridgeless cubic graph contains a perfect matching.
Proof [apply Tutte's thm]

Theorem Every cubic graph with at most two bridges contains a perfect matching.

Theorem 3.7 Let G be a connected graph of order $n \geq 3$. If $deg(u) + deg(v) \geq n$ for all $u \nsim v$, then G is Hamiltonian.
Proof [construct maximal counterexample]

Theorem The Petersen graph is not Hamiltonian.

Theorem There do not exist two disjoint perfect matchings in the Petersen graph.
Proof [by contradiction, their union is 2-regular and thus a union of disjoint even cycles.]

Theorem No tree has two distinct perfect matchings.
Proof. [Suppose a tree has two distinct perfect matching and consider their symmetric difference.]

Lemma Let M_1, M_2 be two distinct perfect matchings in a graph G. Then every vertex in $G[M_1 \Delta M_2]$ has degree 2.

Theorem 3.10 Let G be a connected graph of order n. Assume there exists $u, v \in V$, $u \nsim v$, and $deg(u) + deg(v) \geq n$. Then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.

Theorem 3.13 A graph is Hamiltonian if and only if its closure is Hamiltonian.

Theorem Let G be a cubic graph with at most two bridges. Then G contains a perfect matching.
Proof [by Tutte's thm, counting]

Theorem Q_n is Hamiltonian for all $n \geq 2$.
Proof [by induction on n]

Theorem If G is Hamiltonian then $comp(G - S) \leq |S|$ for all $S \subseteq V$.

Corollary A graph with a cut-vertex is not Hamiltonian.

Euler Identity (Theorem 5.1) For every connected plane graph of order n, size m, with r regions, $n - m + r = 2$.

Generalized Euler Identity For a disconnected plane graph with k components, $n - m + r = k + 1$.
Proof. Let each component have order v_i, size m_i and r_i regions. By Euler Id, for each component we have $n_i - m_i + r_i = 2$. Summing these and noting that $\sum n_i = n$, $\sum m_i = m$, and $\sum r_i = r + k - 1$, we have $n - m + (r + k - 1) = 2k$, and thus $n - m + r = k + 1$.

Theorem For a maximal planar graph of order $n \geq 3$ and size m, $m = 3n - 6$.
Proof. Since G is maximal, it is connected, and each face has degree 3, so the total degree of the faces is $3r = 2m$. And by Euler Id, $r = 2 - n + m$. Then $3(2 - n + m) = 2m$ and so $m = 3n - 6$.

Corollary (5.2) For a planar graph of order $n \geq 3$ and size m, $m \leq 3n - 6$.

Corollary 5.3 Every complete graph K_n of order $n \geq 5$ is nonplanar.

Corollary 5.4 Every planar graph contains a vertex of degree 5 or less. That is, if G is planar, then $\delta(G) \leq 5$.

Corollary 5.5 The graph $K_{3,3}$ is nonplanar.

Theorem 5.6 If G is a maximal planar graph of order 4 or more, then the degree of every vertex of G is at least 3. That is, $\delta(G) \geq 3$.

Theorem 5.12 For a plane graph G of order n with Hamiltonian cycle C, with r_i the number of regions interior to C whose boundary contains exactly i edges, and r'_i the number of regions exterior to C whose boundary contains exactly i edges,

$$\sum_{i=3}^{n}(i-2)(r_i - r'_i) = 0.$$

Kuratowski's Theorem (5.14) A graph is planar if and only if it contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.

Corollary The Petersen graph is nonplanar.

Wagner's Theorem (5.17) A graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G.

Theorem If G is a connected regular planar graph of order n, and with n regions in any planar embedding of G, then $G \cong K_1$ or $G \cong K_4$.
Proof. [Application of Euler Id and First Thm.]

Theorem If G is a maximal planar graph of order n at least 3, and with n regions in any planar embedding of G, then $G \cong K_4$.
Proof. [Application of Euler Id and First Thm.]

Theorem If G is a graph of order n at least 6 that contains three spanning trees T_1, T_2, T_3 such that every edge of G belongs to exactly one of the these trees, then G is nonplanar.
Proof. [Application of $E = V - 1$ and $m \leq 3n - 6$.]

Theorem If G is a nontrival planar graph of order n whose complement is a spanning tree, then n is 7.
Proof. [Application of $E = V - 1$, $m = 3n - 6$.]

Theorem 6.1 If H is a subgraph of G then $\chi(H) \leq \chi(G)$.

Corollary 6.2 For every graph G, $\omega(G) \leq \chi(G)$.

Proposition 6.3 For graphs $G_1, G_2, \ldots, G_k$ and $G = G_1 \cup G_2 \cup \ldots G_k$,
$$\chi(G) = max \ \ \chi(G_i)$$

Corollary 6.4 If G is a graph with components $G_1, G_2, \ldots, G_k$, then
$$\chi(G) = max \ \ \chi(G_i)$$

Proposition 6.6 For graphs $G_1, G_2, \ldots, G_k$ and $G = G_1 + G_2 + \ldots G_k$,
$$\chi(G) = \sum \chi(G_i)$$

Proposition 6.7 $\chi(G) \leq 2$ if and only if G is bipartite.

Lemma If $\chi(G) \geq 3$ then G contains an odd cycle.

Theorem 6.10 If G is graph of order n, then
$$\frac{n}{\alpha(G)} \le \chi(G) \le n - \alpha(G) + 1.$$

Proof. First, we show that $\frac{n}{\alpha} \le \chi$. Let $\{V_i\}_{1\le i\le \chi}$ be a partition of the vertices. Then $|V_i| \le \alpha\ \forall i$ and so $n = \sum |V_i| \le \alpha\chi$. Second, we show that $\chi \le n + 1 - \alpha$. Let A be a maximum independent set of vertices. Color all of the vertices in A a single color, and each of the $n - \alpha$ vertices in $G - A$ a unique color. This is a $(n - \alpha + 1)$-coloring.

Theorem If $\frac{n}{\alpha(G)} = \chi(G) = n - \alpha(G) + 1$, then $G \cong K_n$ or $G \cong \overline{K_n}$.
Proof. $\frac{n}{\alpha} = n - \alpha + 1 \implies n = \alpha n - \alpha^2 + \alpha \implies \alpha^2 - \alpha(n+1) + n = 0 \implies (\alpha - 1)(\alpha - n) = 0 \implies \alpha = 1, n$. If $\alpha = 1$, then $G \cong K_n$, and if $\alpha = n$, then $G \cong \overline{K_n}$.

Theorem 6.16 For any Kneser graph $KG(m, n)$,
$$\chi(KG(m, n)) = m - 2n + 2.$$

Theorem $Kneser(5, 2) \cong Petersen.$

Mycielski's Theorem (6.17) For any integer $n \ge 2$, there exists a graph G such that $\omega(G) = 2$ (that is, G is triangle-free) and $\chi(G) = n$.

Theorem Let S be a color class of a k-coloring of a k-chromatic graph G, where $k \ge 2$. Then there is a component H of $G - S$ such that $\chi(H) = k - 1$.

Theorem If $k > 2$ and G is a k-colorable graph of order n such that $\delta(G) > \left(\frac{k-2}{k-1}\right) n$, then G is k-chromatic.